International Journal Multidisciplinary
Homepage : https://journal.antispublisher.com/index.php/IJMI
Email : admin@antispublisher.com

e-ISSN : 3031-9870
IJMI, Vol. 3, No. 2, May 2026
Page 300-315

Solving Two-dimensional Time-Space Fractional diffusion Problem by using Least-Squares-Petrov-Galerkin Method

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DOI : https://doi.org/10.61796/ijmi.v3i2.459

Sections Info

Article history:
Submitted: Desember 07, 2025
Final Revised: January 21, 2026
Accepted: February 18, 2026
Published: March 10, 2026

Keywords:
Two-dimensional time-space fractional diffusion problem
Least-squares method
Petrov-Galerkin method
Caputo-Fabrizio fractional derivative
Orthogonal polynomials

ABSTRACT

Objective: *Nanotechnology (nano: One billionth) is a novel arena with promising applications in the field of medicine, especially pharmaceuticals for safe and targeted drug delivery. The skin is a phenomenal tool for investigation of nanocarriers for drug delivery for topical and dermatological application.* ***Method:*** *The physicochemical characteristics of the nanoparticles, such as rigidity, hydrophobicity, size and charge are crucial to the skin permeation mechanism. Even though different ways have been found to make therapeutics penetrate and permeate the skin better, they are often harsh and could cause permanent damage to the stratum corneum.* ***Results:*** *Nanosized carrier systems offer a novel methodology for contemporary technologies, causing negligible disruption to the skin's natural barrier function.* ***Novelty:*** *This Review discusses the use of Nano carriers to deliver drug molecules, genetic material, and vaccines into the skin, emphasizing nanotoxicology research and recent clinical advancement for market translation.*

INTRODUCTION

Fractional Calculus (FC) is a fundamental concept in applied mathematics. It studies instantaneous and continuous variable changes to explain complex natural phenomena. In recent decades, scientists, engineers, and applied mathematicians have increasingly focused on fractional calculus. Its ability to model real-world processes with memory and hereditary effects has broadened its use. Fractional models now appear in physics, biology, engineering, hydrology, fluid mechanics, viscoelasticity, and finance [1], [2], [3], [4], [5].

Differential equations containing fractional-order derivatives offer more flexibility compared to differential equations of integer order. Due to their non-local properties, fractional derivatives are widely employed in mathematical and physical modeling. The growing range of applications has motivated researchers to study both the analytical and numerical solutions of fractional differential problems (FDPs). In many cases, obtaining an analytical solution is either impossible or extremely challenging, leading to the development of efficient numerical techniques. Among these, the spectral method, the finite difference method, the finite element method, and the Chebyshev wavelet approach are frequently used [6], [7], [[8], [9], [10], [11].

Fractional models have been described by authors using the Caputo or RL derivatives. However, the unique kernel of these variations is a weakness. A non-local derivative has been proposed by Caputo and Fabrizio to address this problem [12]. This suggests that the exponential kernel of the Caputo-Fabrizio derivative (CF) is non-

singular. The fact that non-singular kernels behave better than singular kernels, which can result in improved convergence qualities and more accurate solutions, is one benefit of employing them in fractional calculus. Furthermore, some physical phenomena, including diffusion, can be described more naturally and practically by non-singular kernels. This is what inspired several scholars to use it to model numerous occurrences and solve numerous equations in a variety of scientific fields [13], [14]. This study focuses on the two-dimensional time-space fractional diffusion problem (2D-TSFDP) in conjunction with the fractional CF derivative. 2D-TSFDP have attracted considerable attention in recent years for their ability to describe anomalous diffusion phenomena with greater accuracy than classical diffusion models. This is because these models replace integer derivatives with fractional derivatives, enabling the consideration of memory and nonlocal effects in modeling many physical and engineering phenomena. For instance, A finite difference scheme for TSFDP with Dirichlet and fractional boundary conditions, where the time-fractional derivative is defined in the Caputo sense and the space-fractional derivative in the Riemann-Liouville sense, has been developed by Xie et al. [15]. Pandey et al. [16] proposed an iterative method to solve the two-dimensional time-space fractional advection-reaction-diffusion problem. Their approach uses a homotopy perturbation method to construct approximate solutions, combining the Laplace transform for time discretization with Caputo fractional-order derivatives to model memory effects and anomalous diffusion. Farnam et al. [17] proposed a numerical scheme for the approximate solution of 2D-TSFDP based on a second-order discretization of the time-fractional derivative, combined with the Chebyshev spectral collocation method of the third kind to approximate the spatial variables. Kittisopaporn et al. [18] presented A finite difference scheme based on the Grünwald–Letnikov approximation to solve 2D-TSFDP with Caputo time derivative and Riemann-Liouville space derivatives, leading to a sparse linear system solved using gradient-descent iterative techniques. Wang and Cai [19] employed a fast finite difference scheme based on the L2-1σ formula and fractional centered differences to solve 2D-TSFDP with second-order accuracy in both time and space.

The main objective of this paper is to obtain an approximate solution for 2D-TSFDP by employing Laguerre polynomial as trial functions and Chebyshev polynomial as test functions in conjunction with the Least squares method (LSM) and Petrov-Galerkin (PG) technique. Consider 2D-TFDP following form [20].

$$ {}_0^{CF}D_t^{\alpha}\Phi(x,y,t) = {}_0^{CF}D_x^{\beta}\Phi(x,t) \ + {}_0^{CF}D_y^{\gamma}\Phi(x,y,t) \ + f(x,y,t), \qquad (1) $$

under the initial condition:

$$ \Phi(x,y,0) = \psi(x,y); \quad (x,y) \in \Omega, \qquad (2) $$

and the boundary conditions:

$$ \Phi(x,y,t) = 0 \qquad \text{for } (x,y,t) \in \partial\Omega \qquad (3) $$

where $0 < \alpha < 1, 1 < \beta, \gamma < 2,\ (x,y) \in \Omega,\ \Omega = ([0,1] \times [0,1]), 0 < t < T\ \ \psi(x)$ is a given smooth function, $f(x,y,t)$ is the source term, the symbols ${}_0^{CF}D_t^{\alpha}$ denote the CF time fractional differentiation of order α, ${}_0^{CF}D_x^{\beta}$, ${}_0^{CF}D_y^{\gamma}$ are the CF space fractional differentiation of order β, γ respectively, and T is the final time. The motivation for this work comes from

the need to present simple and effective numerical technique to solve 2D-TSFDP. The Least-square Petrov-Galerkin method (LSPGM) is adopted because it can achieve high accuracy with a small number of collocation points. Although the promising results, there are certain limitations to the suggested method while LSPGM is exhibits high accuracy for solving 2D-TSFDP, it can become computationally expensive when applied to situations that are more complicated or have more dimensions. The major advantage of this study is that introduces a highly efficient and accurate numerical approach for solving 2D-TSFDP. The novelty of this paper lies in the application of LSPGM to solve the 2D-TSFDP involving the CF fractional derivative. This strategy yields highly accurate numerical results while requiring fewer collocation points compared with traditional methods. Consequently, the present work makes a notable contribution to field of fractional differential equations by introducing a robust and efficient numerical framework for solving 2D-TSFDPs, we propose a new method that combines method and least-Squares-Petrov-Galerkin method to solve the fractional (1). By combining least-squares method with PG method with via Chebyshev polynomials as test functions and Laguerre polynomials as trial functions and CF fractional derivative, this study introduces a novel technique well suited for issues such as diffusion problems.

The principal sections of this paper are organized as follows: Section 2 presents the fundamental definitions of the fractional derivatives. In section 3 the algorithm LSPGM is applied to solve 2D-TSFDP. Section 4 discusses an error analysis. In section 5, presents some numerical examples are verify the accuracy and efficiency of the proposed method. Finally, conclusions will be given in Section 6.

RESEARCH METHOD

This section revisits basic definitions about fractional derivatives, that are relevant to study.

Definitions [21]

The definition for the gamma function, denotes by Γ(z), is as follows:

$$\Gamma(z) = \int_0^{\infty} \tau^{z-1} e^{-\tau} d\tau, \quad z \in \mathbb{C}, Re(z) > 0.$$

The Riemann-Liouville fractional derivative of order α of $f(t)$ for $\alpha > 0, n = [\alpha] + 1$ is defined as:

$${}^{RL}_{a}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \displaystyle\int_a^t f(\tau)\,(t-\tau)^{(n-\alpha-1)} d\tau, & \alpha \in (n-1, n], n \in \mathbb{N} \\ f(t), & \alpha = 0 \end{cases}$$

where[α]is the greatest integer function.

Let $0 < \alpha < 1$. The Caputo fractional derivative of order α of $f(t)$ is defined as:

$${}^{C}_{a}D_t^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_a^t f^{(n)}(\tau)\,(t-\tau)^{(n-\alpha-1)} d\tau, & \alpha \in (n-1, n], \ n \in \mathbb{N} \\ f(t), & \alpha = n \end{cases}$$

Definition [12]

Let $\alpha \in [0,1)$. The fractional derivative with the Caputo-Fabrizio operator ${}_a^{CF}D_t^{\alpha} f(t)$ of order α, for a function $f(t) \in H^1(a,b), a < b$ is defined as:

$$ {}_0^{CF}D_t^{\alpha} f(t) = \begin{cases} \dfrac{\varepsilon(\alpha)}{1-\alpha}\displaystyle\int_0^t \exp\left(\frac{-\alpha(t-\tau)}{1-\alpha}\right) f'(\tau)\, d\tau, & 0 \le \alpha < 1 \\ \dfrac{df(t)}{dt}, & \alpha = 1 \end{cases} $$

where $f'(\tau)$ is derivative of u and $\varepsilon(\alpha)$ is a normalization function and satisfies the conditions $\varepsilon(0) = \varepsilon(1) = 1$.

The LSPGM algorithm

In this section, we present the new LSPGM, which employs orthogonal polynomials to construct an approximate solution for the 2D-TFDP described in Eqs. (1) and (2). Specifically, the Laguerre polynomials are used as the trial basis functions, while the Chebyshev polynomials are utilized as the test basis functions [22, 23]. One of the principal benefits of the PG method is its flexibility in choosing the test functions independently of the trial basis. This allows us to adopt a suitable set of test functions that differs from the basis functions, thereby enhancing the stability and accuracy of the resulting approximation. The main steps of the algorithm of the proposed method can be summarized as follows:

Now, define

$$ \mathbb{P}_{nme} = span\{\varphi_{ijk}(x,y,t);\ \ i = 0,1,\dots,n, j = 0,1,\dots,m, k = 0,1,\dots,e\} \tag{4} $$

$$ V_{nme} = \{\Phi(x,y,t) \in \mathbb{P}_{nme} : \Phi(x,y,t) = 0,\} \\ ,\Phi(x,y,0) = \psi(x,y) $$

where $\mathbb{P}_{nme}$ is the space of all polynomials defined in Ω of degree at most nmk and $\varphi_i(x), \varphi_j(y)\varphi_k(t)$ are Laguerre polynomials, for each $i = 0,1,\dots,n, j = 0,1,\dots,m, k = 0,1,\dots,e$.

Step 1: To find the approximate solution $\Phi_{nme}(x,y,t) \in V_{nme}$ of Eq. (1), apply PGM as:

$$ \left({}_0^{CF}D_t^{\alpha}\Phi_{nme}(x,y,t), v_{nme}(x,y,t)\right)_{\omega} = \left({}_0^{CF}D_x^{\beta}\Phi_{nme}(x,y,t), v_{nme}(x,y,t)\right)_{\omega} + \left({}_0^{CF}D_y^{\gamma}\Phi_{nme}(x,y,t), v_{nme}(x,y,t)\right)_{\omega} + \left(f(x,y,t), v_{nme}(x,y,t)\right)_{\omega}, \quad (3.43) $$

(5)

$\forall\, v_{nme} \in \mathbb{P}_{nme}$, where

$$ \left({}_0^{CF}D_t^{\alpha}\Phi_{nme}(x,y,t), v_{nme}(x,y,t)\right) = \int_0^1\int_0^1\int_0^1 {}_0^{CF}D_t^{\alpha}\Phi_{nme}(x,y,t) v_{nme}(x,y,t)\omega(x,y,t)dxdydt $$

$$ \left({}_0^{CF}D_x^{\beta}\Phi_{nme}(x,y,t), v_{nme}(x,y,t)\right)_{\omega} = \int_0^1\int_0^1\int_0^1 {}_0^{CF}D_x^{\beta}\Phi_{nme}(x,y,t) v_{nme}(x,y,t)\omega(x,y,t)dxdydt $$

$$\left({}_0^{CF}D_y^{\gamma}\Phi_{nme}(x,y,t), v_{nme}(x,y,t)\right)_{\omega} = \int_0^1\int_0^1\int_0^1 {}_0^{CF}D_y^{\gamma}\Phi_{nme}(x,y,t)v_{nme}(x,y,t)\omega(x,y,t)dxdydt,$$

and,

$$\left(f(x,y,t), v_{nme}(x,y,t)\right)_{\omega} = \int_0^1\int_0^1\int_0^1 f(x,y,t)v_{nme}(x,y,t)\omega(x,y,t)dxdydt$$

The approximate solution given as follows:

$$\Phi_{nme}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e} c_{ijk}\left(\varphi_i(x)\varphi_j(y)\varphi_k(t) + \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\right). \quad (6)$$

where c_{ijk} are unknown constants coefficients and β_{ijk} are auxiliary constants.

Step 2: Put (6) into (5), to get:

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e} c_{ijk}\big({}_0^{CF}D_t^{\alpha}(\varphi_i(x)\varphi_j(y)\varphi_k(t) &+ \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\ &= \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e} c_{ijk}\big({}_0^{CF}D_x^{\beta}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\ &+ \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\ &+ \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e} c_{ijk}\big({}_0^{CF}D_y^{\gamma}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\ &+ \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\ &+ (f, \phi_r(x)\phi_s(y)\phi_l(t))_{\omega}\ , \end{aligned} \quad (7)$$

where $\emptyset_r(x), \emptyset_s(y), \phi_l(t), r = 0,1,\cdots,n, s = 0,\dots,m, l = 0,1,\dots,k$ are a Chebyshev polynomial test function with its weight function $\omega(x,y,t) = \frac{1}{\sqrt{(1-x^2)(1-y^2)(1-t^2)}}$.

Step 3: From (7) can be written the residual $R_{nm}(x,y;\ c_{ij})$ as follows:

$$\begin{aligned} R_{nme}(x,y,t;\ c_{ijk}) &= \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e} c_{ijk}\big({}_0^{CF}D_t^{\alpha}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\ &+ \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\ &- \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e} c_{ijk}\big({}_0^{CF}D_x^{\beta}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\ &+ \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\ &- \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e} c_{ijk}\big({}_0^{CF}D_y^{\gamma}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\ &+ \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\ &- (f, \phi_r(x)\phi_s(y)\phi_l(t))_{\omega}\ , \end{aligned} \quad (8)$$

Step 4: From Eq. (8) will be found Least-square method as:

$$S(c_{000},\dots,c_{nme}) = \int_0^1\int_0^1\int_0^1 \left(R_{nme}(x,y,t;\ c_{ijk})\right)^2 \omega_1(x,y,t)dxdydt, \quad (9)$$

where $\omega_1(x,y,t)$ denotes the positive weight function, for simplification will be chosen $\omega_1(x,y,t)=1$.

Therefore, Eq. (8) becomes

$$
\begin{aligned}
&S(c_{000},\dots,c_{nme})\\
&=\int_0^1\int_0^1\int_0^1\Bigg\{\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}\big({}^{CF}_{0}D_t^{\alpha}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\
&+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\big),\phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\
&-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}\big({}^{CF}_{0}D_x^{\beta}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\
&+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\big),\phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\
&-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}\big({}^{CF}_{0}D_y^{\gamma}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\
&+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\big),\phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\
&-(f,\phi_r(x)\phi_s(y)\phi_l(t))_{\omega}\Bigg\}^2 dxdydt. \qquad (10)
\end{aligned}
$$

Step 5: To minimize Eq. (10), the values $c_{ijk}, i=0,1,\dots,n, j=0,1,\dots,m, k=0,1,\dots,e$ can be obtained by minimizing values of S as:

$$\frac{\partial S}{\partial c_{ijk}}=0,\ \ i=0,1,\dots,n, j=0,1,\dots,m, k=0,1,\dots,e\,. \qquad (11)$$

Now, applying Eq. (11) in Eq. (10) yields

$$
\begin{aligned}
&\int_0^1\int_0^1\int_0^1\Bigg[\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}\big({}^{CF}_{0}D_t^{\alpha}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\
&+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\big),\phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\
&-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}\big({}^{CF}_{0}D_x^{\beta}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\
&+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\big),\phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\
&-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}\big({}^{CF}_{0}D_y^{\gamma}(\varphi_i(x)\varphi_j(y)\varphi_k(t)\\
&+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\big),\phi_r(x)\phi_s(y)\phi_l(t)\big)_{\omega}\\
&-(f,\phi_r(x)\phi_s(y)\phi_l(t))_{\omega}\Bigg]
\end{aligned}
$$

$$\times\left[\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}({}_0^{CF}D_t^{\alpha}(\varphi_i(x)\varphi_j(y)\varphi_k(t)+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)),\phi_r(x)\phi_s(y)\phi_l(t))_{\omega}\right.$$
$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}\left({}_0^{CF}D_y^{\gamma}\left(\varphi_i(x)\varphi_j(y)\varphi_k(t)\right.\right.$$
$$\left.+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\right),\phi_r(x)\phi_s(y)\phi_l(t)\Big)_{\omega}$$
$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}\left({}_0^{CF}D_y^{\gamma}\left(\varphi_i(x)\varphi_j(y)\varphi_k(t)\right.\right.$$
$$\left.\left.+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\right),\phi_r(x)\phi_s(y)\phi_l(t)\Big)_{\omega}\right]dxdydt$$
$$=0, \qquad (12)$$

Step 6: The algebraic system produced from Eq. (12) consist of $(n+1)\times(m+1)\times(e+1)$ equations in $(n+1)\times(m+1)\times(e+1)$ unknown coefficients $c_{ijk}, i=0,1,\dots,n. j=0,1,\dots,m,\ k=0,1,\dots,e$ is as Eq. (1):

$$AC=F \qquad (13)$$

furthermore:

$$A=\left[A_{ijk}\right]_{i,j,k=0}^{n,m,e},\quad C=[c_{ijk}]_{i,j,k=0}^{nme},F=[f_{rsl}]_{r,s,l=0}^{nme}$$

where

$$A_{ijk}=\left(a_{ijkrsl}\right)_{i,j,k,s,r,l=0}^{nme},\qquad a_{ijkrsl}=\int_0^1\int_0^1\int_0^1 R_{nme}(x,y,t;c_{ijk})h_{rsl}dxdydt,$$
$$f_{rsl}=\int_0^1\int_0^1\int_0^1(f,\phi_r(x)\phi_s(y)\phi_l(t))_w h_{rsl}dx\,dydt$$

and

$$R_{nme}(x,y,t;c_{ijk})$$
$$=\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}({}_0^{CF}D_t^{\alpha}(\varphi_i(x)\varphi_j(y)\varphi_k(t)$$
$$+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)),\phi_r(x)\phi_s(y)\phi_l(t))_{\omega}$$
$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}({}_0^{CF}D_x^{\beta}(\varphi_i(x)\varphi_j(y)\varphi_k(t)$$
$$+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)),\phi_r(x)\phi_s(y)\phi_l(t))_{\omega}$$
$$-\sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}c_{ijk}\left({}_0^{CF}D_y^{\gamma}\left(\varphi_i(x)\varphi_j(y)\varphi_k(t)\right.\right.$$
$$\left.+\beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\right),\phi_r(x)\phi_s(y)\phi_l(t)\Big)_{\omega},$$

$$h_{rsl} = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}({}_0^{CF}D_t^{\alpha}(\varphi_i(x)\varphi_j(y)\varphi_k(t) + \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t))_{\omega} - \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}({}_0^{CF}D_x^{\beta}(\varphi_i(x)\varphi_j(y)\varphi_k(t) + \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)), \phi_r(x)\phi_s(y)\phi_l(t))_{\omega} - \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{e}\left({}_0^{CF}D_y^{\gamma}\left(\varphi_i(x)\varphi_j(y)\varphi_k(t) + \beta_{ijk}\varphi_{i+1}(x)\varphi_{j+1}(y)\varphi_{k+1}(t)\right), \phi_r(x)\phi_s(y)\phi_l(t)\right)_{\omega},$$

the system (12) can be partitioned as

$$\begin{bmatrix} A_{000} & A_{001} & \cdots & A_{00e} \\ A_{010} & A_{011} & \cdots & A_{1me} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n00} & A_{n11} & \cdots & A_{nne} \end{bmatrix}_{(n+1)\times(n+1)\times(e+1)} \begin{bmatrix} c_{000} \\ \vdots \\ c_{nme} \end{bmatrix}_{(n+1)\times(m+1)(e+1)\times 1} = \begin{bmatrix} f_{000} \\ \vdots \\ f_{rsl} \end{bmatrix}_{(n+1)(m+1)\times(e+1)\times 1}, \quad (14)$$

where each block $A_{ir} \in \mathbb{R}^{(m+1)\times(m+1)(e+1)}$, observe that the diagonal blocks in matrix A are square of degree $(n+1)\times(m+1)\times(e+1)$ and non-singular. The Eq. (13) corresponds to a system of $(n+1)\times(m+1)\times(e+1)$ linear algebraic equations with the unknown coefficients orthogonal coefficients $c_{ijk}, i = 0,1,\dots n, j = 0,1,\dots,m, k = 0,1..,e$. Another form of (13) by applying the conditions can be described as

$$[\Phi_{ijk}: c_{ijk}], \qquad i = 0,1,\dots,(n-1), j = 0,1,\dots,(m-1), k = 0,1,\dots,(e-1).$$

where

$$\Phi_{ijk} = [\Phi_{000},\dots,\Phi_{0me},\Phi_{100},\dots,\Phi_{1me},\dots,\Phi_{n00},\dots,\Phi_{nme}]^T,$$

by solving the resulting system of (13), the unknown coefficients can be obtained and consequently, an approximate solution to Eq. (1) is achieved.

RESULTS AND DISCUSSION

Error analysis

This section aims to set an upper bound for the error between the approximate solution $\Phi_{nme}(x,y,t)$ and the exact solution $\Phi(x,y,t)$ in order to assess the correctness and stability of the suggested method

The optimal approximation and interpolation framework, based on orthogonal polynomials, is the basis of the analysis.

Let $\Phi_{nme}(x,y,t) \in V_{nme}$ be the best approximation of $\Phi(x,y,t)$, then the definition of the best approximation enables us to write the following inequality:

$$\|\Phi(x,y,t) - \Phi_{nme}(x,y,t)\|_{\infty} \le \|\Phi(x,y,t) - v_{nme}(x,y,t)\|_{\infty}, \quad (15)$$

$\forall v_{nme}(x,y,t) \in V_{nme}$,

Furthermore, the previous inequality also be true if $v_{nme}(x,y,t)$ denotes the interpolating polynomial for $\Phi(x,y,t)$ at point (x_i,y_j,t_k) where x_i,y_j and t_k are roots of $\varphi_i(x),\varphi_j(y)$ and $\varphi_k(t)$ respectively, where $i=0,1,\dots,n, j=0,1,\dots,m$ and $k=0,1,\dots,e$. By using similar procedures as in [24], we have

$$\begin{aligned}\Phi(x,y,t)-v_{nme}(x,y,t)&=\frac{\partial^{n+1}\Phi(\xi,y,t)}{(n+1)!\,\partial x^{n+1}}\prod_{i=1}^{n}(x-x_i)+\frac{\partial^{m+1}\Phi(x,\eta,t)}{(m+1)!\,\partial y^{m+1}}\prod_{j=1}^{m}(y-y_j)\\&+\frac{\partial^{e+1}\Phi(x,y,\mu)}{(e+1)!\,\partial t^{e+1}}\prod_{k=1}^{e}(t-t_k)\\&-\frac{\partial^{n+m+e+3}\Phi(\bar{\xi},\bar{\eta},\bar{\mu})}{(n+1)!\,(m+1)!\,(e+1)!\,\partial x^{n+1}\partial y^{m+1}\partial t^{e+1}}\prod_{i=1}^{n}(x-x_i)\prod_{j=1}^{m}(y\\&-y_j)\prod_{k=1}^{e}(t-t_k),\end{aligned}\tag{16}$$

where $\xi,\bar{\xi},\eta,\bar{\eta},\mu,\bar{\mu}\in[0,1]$ and will be satisfied

Now, applying max norm we have

$$\begin{aligned}\|\Phi(x,y,t)-\Phi_{nme}(x,y,t)\|_\infty&\le\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{n+1}\Phi(\xi,y,t)}{\partial x^{n+1}}\right|\frac{\|\prod_{i=0}^{n}(x-x_i)\|_\infty}{(n+1)!}\\&+\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{m+1}\Phi(x,\eta,t)}{\partial y^{m+1}}\right|\frac{\left\|\prod_{j=0}^{m}(y-y_j)\right\|_\infty}{(m+1)!}\\&+\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{e+1}\Phi(x,y,\mu)}{\partial t^{e+1}}\right|\frac{\|\prod_{k=0}^{e}(t-t_k)\|_\infty}{(e+1)!}-\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{n+m+e+3}\Phi(\bar{\xi},\bar{\eta},\bar{\mu})}{\partial x^{n+1}\partial y^{m+1}\partial t^{e+1}}\right|\\&\times\frac{\|\prod_{i=0}^{n}(x-x_i)\|_\infty\left\|\prod_{j=0}^{m}(y-y_j)\right\|_\infty\|\prod_{k=0}^{e}(t-t_k)\|_\infty}{(n+1)!(m+1)!(e+1)!},\end{aligned}\tag{17}$$

Since $\Phi(x,y,t)$ is a smooth continuous function on $\Omega=[0,1]^3$, then there are exist three a constant M_1,M_2,M_3 and M_4 such that

$$\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{n+1}\Phi(\xi,t)}{\partial x^{n+1}}\right|\frac{\|\prod_{i=0}^{n}(x-x_i)\|_\infty}{(n+1)!}\le M_1,$$

$$\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{m+1}\Phi(x,\eta)}{\partial y^{m+1}}\right|\frac{\left\|\prod_{j=0}^{m}(y-y_j)\right\|_\infty}{(m+1)!}\le M_2,$$

$$\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{e+1}\Phi(x,y,\mu)}{\partial t^{e+1}}\right|\frac{\|\prod_{k=0}^{e}(t-t_k)\|_\infty}{(e+1)!}\le M_3,$$

$$\max_{(x,y,t)\in\Omega}\left|\frac{\partial^{n+m+3}\Phi(\bar{\xi},\bar{\eta})}{\partial x^{n+1}\partial y^{m+1}\partial t^{e+1}}\right|\frac{\|\prod_{i=0}^{n}(x-x_i)\|_\infty\left\|\prod_{j=0}^{m}(y-y_j)\right\|_\infty\|\prod_{k=0}^{e}(t-t_k)\|_\infty}{(n+1)!\,(m+1)!\,(e+1)!}\le M_4\,, \tag{18}$$

The factor $\|\prod_{i=0}^{n}(x-x_i)\|_\infty$ is minimum value of we can use the one-to-one mapping $x=\frac{1}{2}(z+1)$ between the intervals $[-1,1]$ and $[0,1]$ to conclusion that

$$\begin{aligned}\min_{x_i\in[0,1]}\max_{x\in[0,1]}\left|\prod_{i=0}^{n}(x-x_i)\right| &= \min_{z_i\in[-1,1]}\max_{z\in[-1,1]}\left|\prod_{i=0}^{n}(z-z_i)\right| \\ &= \left(\frac{1}{2}\right)^{n+1}\min_{z_i\in[-1,1]}\max_{z\in[-1,1]}\left|\prod_{i=0}^{n}(z-z)\right|\le \\ &= \left(\frac{1}{2}\right)^{n+1}\min_{z_i\in[-1,1]}\max_{z\in[-1,1]}\left|\frac{\Psi_{n-1}(z)}{\bar{\Psi}_n}\right|, \end{aligned}\tag{19}$$

where $\bar{\Psi}_n=2^n$ is the leading coefficient of $\varphi_{n+1}(z)$ and z_i are the roots of $\Psi_{n-1}(z)$.

Similarly, the factor $\left\|\prod_{j=0}^{m}(y-y_j)\right\|_\infty$ may be minimizes by using the one to one mapping between the intervals $[-1,1]$ and $[0,1]$ to infer that

$$\begin{aligned}\min_{y\in[0,1]}\max_{y\in[0,1]}\left\|\prod_{j=0}^{m}(y-y_j)\right\|_\infty &\min_{u_i\in[-1,1]}\max_{u\in[-1,1]}\left|\prod_{j=0}^{m}(y-y_j)\right| \\ &\le\left(\frac{1}{2}\right)^{m+1}\min_{u_j\in[-1,1]}\max_{u\in[-1,1]}\left|\prod_{j=0}^{m}(u-u_j)\right| \\ &\le\left(\frac{1}{2}\right)^{m+1}\min_{u_j\in[-1,1]}\max_{u\in[-1,1]}\left|\frac{\varsigma_m(u)}{\bar{\varsigma}_m}\right|, \end{aligned}\tag{20}$$

where $\bar{\varsigma}_m=2^{m-2}$ is the leading coefficient of $\varsigma_m(u)=\frac{1+u}{2}\varphi_m(u)$ and u_j are the roots of $\varsigma_m(u)$.

Also, to get the minimum value of the factor $\|\prod_{k=0}^{e}(t-t_k)\|_\infty$, we can use the one-to-one mapping $t=\frac{1}{2}(\bar{t}+1)$ between the intervals $[-1,1]$ and $[0,1]$ to infer that

$$\begin{aligned}\min_{t_i\in[0,1]}\max_{t\in[0,1]}\left\|\prod_{k=0}^{e}(t-t_k)\right\|_\infty &\le\frac{1}{2^{e+1}}\min_{\bar{t}\in[0,1]}\max_{\bar{t}\in[-1,1]}\left\|\prod_{k=0}^{e}(\bar{t}-\bar{t}_k)\right\|_\infty \\ &=\left(\frac{1}{2}\right)^{e+1}\min_{\bar{t}_k\in[-1,1]}\max_{\bar{t}\in[-1,1]}\left|\prod_{j=0}^{m}(\bar{t}-\bar{t}_k)\right|=\left(\frac{1}{2}\right)^{e+1}\min_{\bar{t}_k\in[-1,1]}\max_{\bar{t}\in[-1,1]}\left|\frac{\chi_e(\bar{t})}{\chi_e}\right|, \end{aligned}\tag{21}$$

where $\chi_e=2^{e-2}$ is the leading coefficient of $\chi_e(\bar{t})=\frac{1+\bar{t}}{2}\varphi_e(\bar{t})$ and $\bar{t}_k$ are the roots of $\chi_e(\bar{t})$.

It is known that

$$\max_{z\in[-1,1]}|\Psi_{n-1}(z)| = |\varphi_{n+1}| + |\varphi_{n-1}| = 2,$$

and

$$\max_{u\in[-1,1]}|\varsigma_m(u)| = |\varsigma_m(1)| = 1, \max_{\bar{t}\in[-1,1]}|\chi_e(\bar{t})| = 1 \qquad (22)$$

Therefore, inequalities (18), (19), (20) and (21) contribute to reaching the desired conclusion:

$$\begin{aligned}\|\Phi(x,y,t) - \Phi_{nme}(x,y,t)\|_\infty &\\ \leq \frac{M_1\left(\frac{1}{2}\right)^n}{\Psi_n(n+1)!} + \frac{M_2\left(\frac{1}{2}\right)^{m+1}}{\varsigma_m(u)(m+1)!} + \frac{M_3\left(\frac{1}{2}\right)^{e+1}}{\chi_e(\bar{t})(e+1)!} &\\ + \frac{M_4\left(\frac{1}{2}\right)^{n+m+e+3}}{(n+1)!\,(m+1)!\,(e+1)!}\ , \qquad (23)\end{aligned}$$

Thus, the error bound of the absolute error (AE) is provided by equation (23). The convergence of the suggested method is based on this error bound.

Numerical Examples

This section recounts numerical examples of the 2D-TSFDP to weigh up the method's accuracy and efficiency. Moreover, all calculations were carried out with MATLAB R 2023a software, Table 1 and Table 2.

Example 1: Consider the following 2D-TSFDP of form the [20]:

$${}_0^{CF}D_t^{\alpha}\Phi(x,y,t) = {}_0^{CF}D_x^{\beta}\Phi(x,y,t) + {}_0^{CF}D_y^{\gamma}\Phi(x,y,t) + f(x,y,t),$$

The boundary conditions

$$\Phi(x,0,t) = 0, \Phi(x,1,t) = 0$$
$$\Phi(0,y,t) = 0, \Phi(1,y,t) = 0,$$

and the initial conditions

$$\Phi(x,y,0) = x^2(1-x)^2y(1-y)^2,$$

where $(x,y) \in \Omega$, $\Omega = (0,1)\times(0,1), 0 < t < 1$, and $0 < \alpha < 1, 1 < \beta, \gamma < 2$, the exact solution is $\Phi(x,y,t) = e^{-\rho t}x^2(1-x)^2y^2(1-y)^2, \rho = \frac{\alpha}{1-\alpha}$, and the source function

$$f(x,y,t) = \frac{\rho}{1-\alpha}y^2(y-1)^2x^2(x-1)^2 + \frac{e^{-\rho t}y^2(1-y)^2 24\left(x^{4-\beta}+(1-x)^{4-\beta}\right)}{\Gamma(5-\beta)\cos\left(\frac{\beta\pi}{2}\right)} +$$
$$\frac{e^{-\rho t}}{2\cos\left(\frac{\beta\pi}{2}\right)}y^2(y-1)^2\left(\frac{2x^{2-\beta}+(1-x)^{2-\beta}}{\Gamma(3-\beta)} - \frac{12(x^{3-\beta}+(1-x)^{3-\beta})}{\Gamma(4-\beta)}\right) + \frac{e^{-\rho t}}{2\cos\left(\frac{\beta\pi}{2}\right)}x^2(x-$$
$$1)^2\left(\frac{24(y^{4-\beta}+(1-y)^{4-\beta})}{\Gamma(5-\beta)} - \frac{12(y^{3-\beta}+(1-y)^{3-\beta})}{\Gamma(4-\beta)}\right) + \frac{e^{-\rho t}}{2\cos\left(\frac{\beta\pi}{2}\right)}x^2(x-1)^2\left(\frac{2(y^{2-\beta}+(1-y)^{2-\beta})}{\Gamma(3-\beta)}\right.$$

Table 1. Absolute error and L_2-error for different values of n, m, k, α, β and γ by LSPGM for Example 1.

α	$n = m = e$	***LSPGM***			
		$\beta = 1.2, \gamma = 1.3$		$\beta = 1.8, \gamma = 1.7$	
		Absolute error	L_2**-error**	**Absolute error**	L_2**-error**
0.3	3	1.3851×10^{-6}	1.9335×10^{-7}	5.7811×10^{-6}	7.3610×10^{-7}
	4	1.3202×10^{-7}	3.8187×10^{-8}	6.9123×10^{-7}	1.1624×10^{-7}
	5	9.6743×10^{-8}	3.7432×10^{-8}	1.9448×10^{-7}	8.0132×10^{-8}
	6	1.9597×10^{-8}	1.8533×10^{-8}	1.2020×10^{-7}	5.0715×10^{-8}
0.7	3	2.8450×10^{-6}	1.2615×10^{-6}	1.0120×10^{-5}	2.7636×10^{-6}
	4	1.1831×10^{-6}	5.2580×10^{-7}	4.7426×10^{-6}	9.2319×10^{-7}
	5	5.1043×10^{-7}	1.5646×10^{-7}	7.5865×10^{-7}	1.9836×10^{-7}
	6	2.3621×10^{-7}	2.0494×10^{-7}	6.4115×10^{-7}	4.4034×10^{-7}

Table 2. comparison between the proposed LSPGM and the method in [20] for Example 1.

α	Method in [20] at $n = m = 80$		***LSPGM*** at $n = m = 6$	
0.3	$\beta = 1.2, \gamma = 1.3$	$\beta = 1.8, \gamma = 1.7$	$\beta = 1.3, \gamma = 1.2$	$\beta = 1.8, \gamma = 1.7$
	1.3016×10^{-6}	1.6713×10^{-6}	1.9597×10^{-8}	1.2020×10^{-7}
0.7	3.4094×10^{-7}	2.9480×10^{-7}	2.3621×10^{-7}	6.4115×10^{-7}

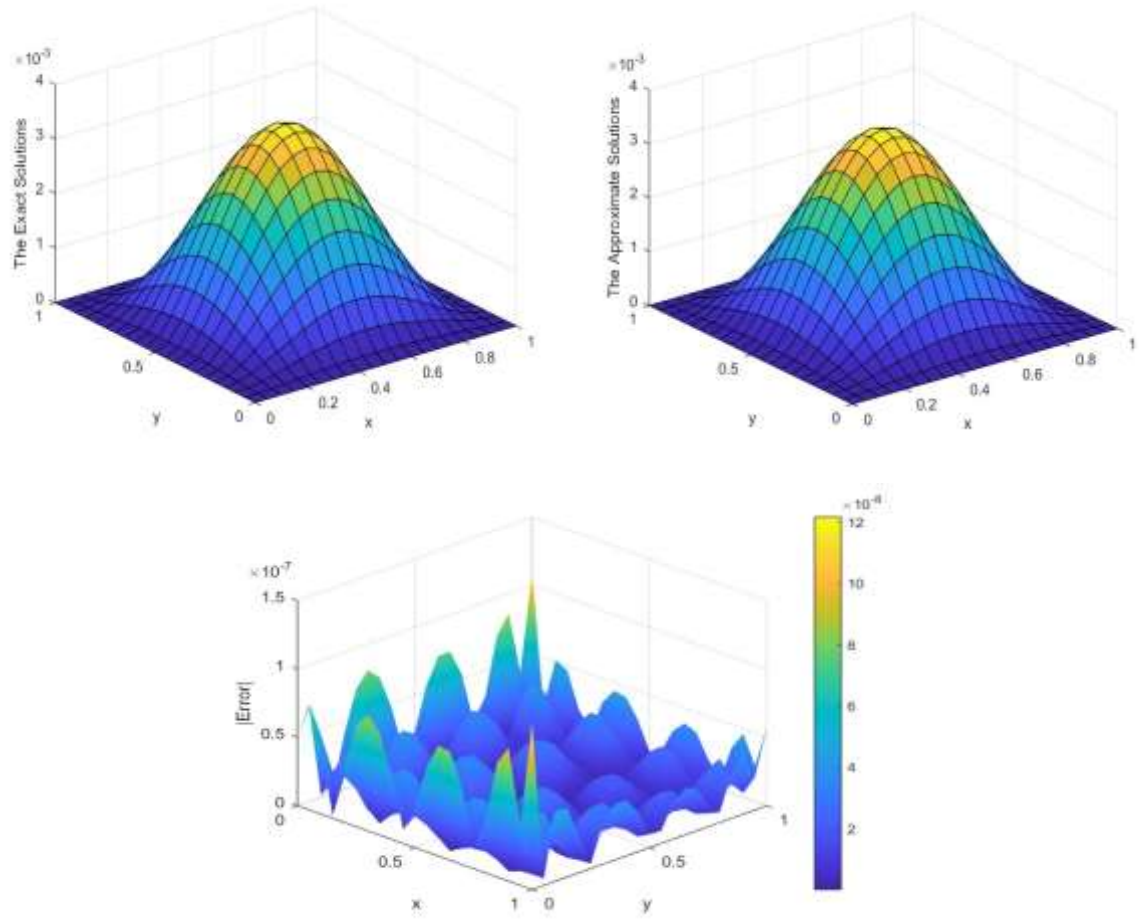

Figure 1. Plot of the exact solution, approximate solutions and AE for Example 4.1 at $\alpha = 0.3,\ \beta, \gamma = 1.2, 1.3$

Table 1, presents the absolute errors obtained by the proposed method at $T = 1$ for different values of n, m, e, α, β and γ. Table 5.2 compares the proposed LSPGM with the technique discussed in [20]. The proposed method achieved lower absolute errors, even for smaller values of $n, m, k,$ than those reported in the cited study, particularly at its maximum value. These results show that the proposed method effectively reduces errors and improves solution accuracy. Figure 1, plots the exact solution, the approximate error, and the absolute error at $\alpha, \beta = 0.3, 0.7, \gamma = 1.2, 1.3$.

CONCLUSION

Fundamental Finding: In this study, we applied new scheme depend on combined the least squares method with Petrov-Galerkin method for developing an efficient numerical approach of two-dimensional time-space fractional diffusion problem in the Caputo-Fabrizio sense. The suggested method takes advantage of orthogonal polynomials, which improves stability, accuracy, and computational efficiency for problems with boundary constraints. The numerical experiments confirmed the effectiveness of this method. All these results confirm that the present method produces accurate solutions even with a relatively small number of basic functions. The development of a new approximate approach LSPGM in the sense CF fractional

derivative to effectively solve 2D-TSFDP. **Implication:** The suggested technique achieves higher accuracy and efficiency using fewer basis functions compared to others methods, this accuracy and efficiency are further demonstrated though a detailed comparison with exact solutions and LSPGM. **Limitation:** The present method focuses on solving the two-dimensional time-space fractional diffusion problem in the Caputo-Fabrizio sense. **Future Research:** Future directions include the extending of the suggested method to solve three-dimensional space-fractional diffusion problems as well as exploring advanced formulations of fractional calculus to enhance performance and applicability.

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